

Numerical semigroups generated by squares, cubes and quartics of three consecutive integers

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Abstract

We derive the polynomial representations for minimal relations of generating set of numerical semigroups $R_n^k = \langle (n-1)^k, n^k, (n+1)^k \rangle$, $k = 2, 3, 4$, $n \geq 3$. We find also the polynomial representations for degrees of syzygies in the Hilbert series $H(z, R_n^k)$ of these semigroups, their Frobenius numbers $F(R_n^k)$ and genera $G(R_n^k)$.

Keywords: Nonsymmetric numerical semigroups, Frobenius number, genus

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1 SYMMETRIC AND NONSYMMETRIC NUMERICAL SEMIGROUPS $\langle (n-1)^k, n^k, (n+1)^k \rangle$

Numerical semigroups $S_3 = \langle d_1, d_2, d_3 \rangle$, generated by three integers, exhibit a non-trivial example of semigroups with well established relations [2] between degrees of syzygies and values of generators. In this regards, the relations $f_k(d_1, d_2, d_3) = 0$, imposed on generators, may increase a number of semigroups with explicitly computable Hilbert series $H(z, S_3)$, Frobenius numbers $F(S_3)$ and genera $G(S_3)$. Usually the generators of such semigroups are represented as elements of some ordered sets: arithmetic [1], almost arithmetic [11] or geometric [10] progressions, Pythagorean triples [7, 2], Fibonacci [9, 3] or Lucas [3] numbers and others.

Recently, the two 3-generated numerical semigroups were treated [8] to establish explicit expressions for their Frobenius numbers. These are semigroups R_n^2 and R_n^3 , generated by squares and cubes of three consecutive integers,

$$R_n^2 = \langle (n-1)^2, n^2, (n+1)^2 \rangle, \quad R_n^3 = \langle (n-1)^3, n^3, (n+1)^3 \rangle, \quad n \geq 3. \quad (1.1)$$

Using the Euclidean algorithm with negative [12] and positive [13] remainders for computing the Frobenius numbers, the authors [8] were able to find polynomial expressions in n for $F(R_n^2)$ and $F(R_n^3)$ on residue class of n modulo 4 and 18,

$$F_j(R_n^2) = \sum_{i=1}^3 A_i^j n^i, \quad j = n \pmod{4}, \quad n \neq 3, 4, 5, 6, 9, 13, \quad A_i^j \in \mathbb{Q}, \quad (1.2)$$

$$F_j(R_n^3) = \sum_{i=1}^5 B_i^j n^i, \quad j = n \pmod{18}, \quad B_i^j \in \mathbb{Q}, \quad n \neq 3, 4, 5, 6, 7, 8, 9, 10, \quad (1.3)$$

11, 18, 26, 27, 36, 45, 54, 63, 72, 90, 108, 126, 144, 162, 180, 198, 216, 234, 252, 270.

A long list of sophisticated formulas for $F_j(R_n^2)$ and $F_j(R_n^3)$ in [8], accompanied by 34 exclusions (6 cases for R_n^2 and 28 cases for R_n^3), poses a question to find another representation (Rep) which allows incorporate all exclusions. Another reason to treat this problem again is to extend it on semigroups R_n^4 and find $H(z, R_n^4)$, $F(R_n^4)$ and $G(R_n^4)$, and to discuss how to deal with a general case of R_n^k , $k \geq 5$.

Note that the excluding values of n in (1.2, 1.3) give rise to the 4 symmetric semigroups, R_3^2 , R_4^2 , R_5^2 and R_3^3 . The rest of 30 semigroups are nonsymmetric.

Simple considerations (Propositions 1, 2) show that no more symmetric numerical semigroups R_n^2 and R_n^3 do exist. To prove that we recall necessary conditions when a semigroup $\langle d_1, d_2, d_3 \rangle$ becomes symmetric,

$$\begin{aligned} a) \quad & d_3 \in \langle d_1, d_2 \rangle, \quad \gcd(d_1, d_2) = 1, \quad d_j > 3, \quad \text{or} \\ b) \quad & \langle d_1, d_2, d_3 \rangle, \quad d_j > 3, \quad \text{satisfies Lemma 1 adapted to three generators,} \end{aligned} \quad (1.4)$$

Lemma 1 [15] *Let a numerical semigroup $\langle d_1, d_2, d_3 \rangle$ be given such that $d_1 = a\delta_1$, $d_2 = a\delta_2$, $\delta_j \in \mathbb{Z}_+$ and $\gcd(\delta_1, \delta_2) = \gcd(a, d_3) = 1$. Then $\langle d_1, d_2, d_3 \rangle$ is symmetric if and only if $d_3 \in \langle \delta_1, \delta_2 \rangle$.*

For semigroups, satisfying Lemma 1, the Frobenius number is given by [6, 5],

$$F(\langle d_1, d_2, d_3 \rangle) = aF(\langle \delta_1, \delta_2 \rangle) + (a-1)d_3, \quad F(\langle \delta_1, \delta_2 \rangle) = \delta_1\delta_2 - \delta_1 - \delta_2. \quad (1.5)$$

1.1 SYMMETRIC NUMERICAL SEMIGROUPS R_n^k

Prove an exclusive property of semigroups R_3^2 , R_4^2 , R_5^2 .

Proposition 1 *There exist only three symmetric semigroups R_n^2 , $n = 3, 4, 5$.*

Proof Note that semigroups R_n^2 , $n = 3, 4$, are symmetric due to requirement (1.4a). Find more n which satisfy (1.4a),

$$(n+1)^2 = a_1(n-1)^2 + a_2n^2, \quad a_1, a_2 \in \mathbb{N}, \quad n > 4. \quad (1.6)$$

Simplifying the last equality we obtain the Diophantine Eq.

$$(a_1 + a_2 - 1)(n-4)^2 + 2(3a_1 + 4a_2 - 5)(n-4) + 9a_1 + 16a_2 - 25 = 0,$$

with constraints (1.6) on three variables a_1, a_2, n which has no solutions.

Consider another way to symmetrize R_n^2 by providing condition (1.4b), which may occur only when $n = 2p + 1$ and results in the Diophantine Eq. in b_1, b_2, p ,

$$(2p + 1)^2 = b_1 p^2 + b_2 (p + 1)^2, \quad b_1, b_2 \in \mathbb{N}, \quad p \geq 2, \quad (1.7)$$

Solving (1.7) as a quadratic equation in p , we get

$$p = \frac{2 - b_2 \pm \Theta}{b_1 + b_2 - 4}, \quad \Theta^2 = b_1 + b_2 - b_1 b_2,$$

Combining the last expression with constraints in (1.7) we find only one appropriate solution, $b_1 = 4, b_2 = 1, p = 2$, which gives rise to semigroup R_5^2 .

The next Propositions deal with symmetric semigroups $R_n^k, k = 3, 4$.

Proposition 2 *There exists only one symmetric semigroup $R_n^3, n = 3$.*

Proposition 3 *There exist only three symmetric semigroups $R_n^4, n = 3, 5, 7$.*

Their proofs are similar to proof of Proposition 1 but much more cumbersome and therefore are given in A.

1.2 NONSYMMETRIC NUMERICAL SEMIGROUPS GENERATED BY THREE INTEGERS

A brief analysis in the previous section focuses us on nonsymmetric semigroups only. In the present paper we calculate the Hilbert series for such semigroups $R_n^k, k = 2, 3, 4$, making use of approach of minimal relations for three generators d_1, d_2, d_3 . Recall this approach following author's article [2].

Let a nonsymmetric numerical semigroup $S_3 = \langle d_1, d_2, d_3 \rangle$, $\gcd(d_1, d_2, d_3) = 1, d_j \geq 3$, be given by matrix of minimal relations, \mathcal{A}_3 , where $a_{ij} \in \mathbb{Z}_+$,

$$\mathcal{A}_3 \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{A}_3 = \begin{pmatrix} a_{11} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} \end{pmatrix}, \quad \begin{cases} \gcd(a_{11}, a_{12}, a_{13}) = 1 \\ \gcd(a_{21}, a_{22}, a_{23}) = 1 \\ \gcd(a_{31}, a_{32}, a_{33}) = 1 \end{cases}, \quad (1.8)$$

$$\begin{aligned} a_{11} &= \min \{v_{11} \mid v_{11} \geq 2, v_{11}d_1 = v_{12}d_2 + v_{13}d_3, v_{12}, v_{13} \in \mathbb{N} \cup \{0\}\}, \\ a_{22} &= \min \{v_{22} \mid v_{22} \geq 2, v_{22}d_2 = v_{21}d_1 + v_{23}d_3, v_{21}, v_{23} \in \mathbb{N} \cup \{0\}\}, \\ a_{33} &= \min \{v_{33} \mid v_{33} \geq 2, v_{33}d_3 = v_{31}d_1 + v_{32}d_2, v_{31}, v_{32} \in \mathbb{N} \cup \{0\}\}. \end{aligned} \quad (1.9)$$

All matrix elements a_{ij} are non-negative integers [6, 2] such that

$$\begin{aligned} a_{11} &= a_{21} + a_{31}, \quad a_{22} = a_{12} + a_{32}, \quad a_{33} = a_{13} + a_{23}, \quad a_{jj} \geq 2, \quad a_{ij} \geq 1, \quad i \neq j, \\ d_1 &= a_{22}a_{33} - a_{23}a_{32}, \quad d_2 = a_{33}a_{11} - a_{31}a_{13}, \quad d_3 = a_{11}a_{22} - a_{12}a_{21}. \end{aligned} \quad (1.10)$$

Then the rational Rep of the Hilbert series $H(z, S_3)$, the Frobenius number $F(S_3)$ and genus $G(S_3)$ are given by formulas [2],

$$\begin{aligned}
H(z, S_3) &= \left(1 - z^{a_{11}d_1} - z^{a_{22}d_2} - z^{a_{33}d_3} + z^{b_{11}} + z^{b_{22}}\right) \prod_{i=1}^3 \left(1 - z^{d_i}\right)^{-1}, \\
b_{11} &= D_0 + D_1, \quad b_{22} = D_0 + D_2, \quad a_{11}d_1 + a_{22}d_2 + a_{33}d_3 = b_{11} + b_{22}, \\
F(S_3) &= \max\{F_1, F_2\}, \quad 2G(S_3) = 1 + D_0 + D_1 + D_2 - D_3, \\
F_1 &= b_{11} - D_3, \quad F_2 = b_{22} - D_3. \\
D_0 &= a_{11}a_{22}a_{33}, \quad D_1 = a_{12}a_{23}a_{31}, \quad D_2 = a_{13}a_{32}a_{21}, \quad D_3 = d_1 + d_2 + d_3.
\end{aligned} \tag{1.11}$$

Based on (1.11) we reduce a large number of exclusive semigroups [8] with $F(S_3)$ which differ from polynomials (1.2). This exclusion may happen when in different ranges of n a difference $D_1(n) - D_2(n)$ may change its sign. In other words, the both sequences $F_1(n)$ and $F_2(n)$ contribute to the polynomial Rep of $F(S_3)$.

2 NUMERICAL SEMIGROUPS R_n^2 , $n \geq 6$

Write the third relation in (1.9) for R_n^2 , $a_{33}(n+1)^2 = a_{32}n^2 + a_{31}(n-1)^2$, i.e.,

$$a_{32}n^2 = (a_{33} - a_{31})(n^2 + 1) + 2(a_{33} + a_{31})n.$$

Choose $a_{33} = a_{31}$ that results in $a_{32} = 4a_{33}/n$. The whole matrix \mathcal{A}_3 satisfies relations (1.8),

$$\begin{pmatrix} a_{21} + a_{33} & 4a_{33}/n - a_{22} & a_{23} - a_{33} \\ -a_{21} & a_{22} & -a_{23} \\ -a_{33} & -4a_{33}/n & a_{33} \end{pmatrix}, \quad \begin{aligned} (n-1)^2 &= a_{33}(a_{22} - 4a_{23}/n), \\ n^2 &= a_{33}(a_{21} + a_{23}), \\ (n+1)^2 &= a_{33}(a_{22} + 4a_{21}/n). \end{aligned}$$

In order to provide all entries in \mathcal{A}_3 be integers we consider four different cases, $n = j \pmod{4}$.

1. $n = 4m$, $a_{33} = a_{31} = m$, $a_{32} = 1$.

$$\begin{pmatrix} a_{21} + m & 1 - a_{22} & a_{23} - m \\ -a_{21} & a_{22} & -a_{23} \\ -m & -1 & m \end{pmatrix}, \quad \begin{aligned} (4m-1)^2 &= a_{22}m - a_{23}, \\ (4m+1)^2 &= a_{22}m + a_{21}. \end{aligned} \tag{2.1}$$

Two Eqs. (2.1) for a_{21} , a_{22} , a_{23} allow to choose the following parameterization,

$$a_{21} = km + 1, \quad a_{22} = 16m + 8 - k, \quad a_{23} = (16 - k)m - 1, \quad 1 \leq k \leq 15,$$

$$\begin{pmatrix} (k+1)m + 1 & -(16m + 7 - k) & -[(k-15)m + 1] \\ -(km + 1) & 16m + 8 - k & -[(16 - k)m - 1] \\ -m & -1 & m \end{pmatrix}.$$

Since $a_{13}, a_{23} \geq 1$ there is only one solution $k = 15$, $m \geq 2$, that gives

$$\begin{pmatrix} 16m+1 & -8(2m-1) & -1 \\ -(15m+1) & 16m-7 & -(m-1) \\ -m & -1 & m \end{pmatrix}, \begin{array}{l} G = 4m(34m^2 - 21m + 2), \\ F = 20, \quad m = 1, \\ F = 272m^3 - 168m^2 + m - 2, \quad m \geq 2. \end{array}$$

For the rest semigroups we skip the parameterization details of a_{2j} and give the final formulas.

2. $n = 4m + 2$, $a_{33} = a_{31} = 2m + 1$, $a_{32} = 2$.

$$\begin{pmatrix} 9m+5 & -(8m-1) & -(m+1) \\ -(7m+4) & 8m+1 & -m \\ -(2m+1) & -2 & 2m+1 \end{pmatrix}, \begin{array}{l} G = m(80m^2 + 71m + 16), \\ F = 312, \quad m = 1, \\ F = 160m^3 + 128m^2 + 10m - 9, \quad m \geq 2. \end{array}$$

3. $n = 4m + 1$, $a_{33} = a_{31} = 4m + 1$, $a_{32} = 4$.

$$\begin{pmatrix} 7m+2 & -4(m-1) & -(3m+1) \\ -(3m+1) & 4m & -m \\ -(4m+1) & -4 & 4m+1 \end{pmatrix}, \begin{array}{l} G = 2m(32m^2 + 9m + 1), \\ F = 128m^3 - 20m - 5, \quad m \geq 4 \\ F = 112m^3 + 48m^2 + 8m - 1, \quad m \leq 3. \end{array}$$

4. $n = 4m + 3$, $a_{33} = a_{31} = 4m + 3$, $a_{32} = 4$.

$$\begin{pmatrix} 5m+4 & -4m & -(m+1) \\ -(m+1) & 4(m+1) & -(3m+2) \\ -(4m+3) & -4 & 4m+3 \end{pmatrix}, \begin{array}{l} G = 2(32m^3 + 57m^2 + 33m + 6), \\ F = 128m^3 + 224m^2 + 124m + 19, \quad m \geq 1. \end{array}$$

The above formulas for $F(R_n^2)$ coincide with those obtained in [8] when $b_{22} > b_{11}$. In the opposite case ($b_{22} < b_{11}$) we arrive at the other formulas, e.g., as in the case $n = 4m + 1$, $m \leq 3$. There exist 3 exceptional symmetric semigroups R_3^2 , R_4^2 , R_5^2 , with minimal relations which do not obey the matrix Reps presented above.

3 NUMERICAL SEMIGROUPS R_n^3 , $n \geq 4$

Write the third relation in (1.9) for R_n^3 ,

$$a_{32}n^3 = (a_{33} - a_{31})(n^3 + 3n) + (a_{33} + a_{31})(3n^2 + 1). \quad (3.1)$$

Choose the Rep $a_{31} = (pn + q)$, $a_{33} = (pn - q)$, $p, q \in \mathbb{Q}$, and insert it into (3.1),

$$a_{32}n^2 = 2p(3n^2 + 1) - 2q(n^2 + 3), \quad \text{or} \quad a_{32} = 6p - 2q + 2(p - 3q)/n^2.$$

To eliminate the dependence of a_{32} on n^{-2} in the last relation we put

$$p = 3q, \quad a_{31} = q(3n + 1), \quad a_{32} = 16q, \quad a_{33} = q(3n - 1).$$

To satisfy $\gcd(a_{31}, a_{32}, a_{33}) = 1$ in (1.9), we have to distinguish two different cases: $q = 1$ if $n = 2N$ and $q = 1/2$ if $n = 2N + 1$.

3.1 NUMERICAL SEMIGROUPS R_n^3 , $n \equiv 0 \pmod{2}$

The matrix of minimal relations \mathcal{A}_3 reads,

$$\begin{pmatrix} a_{21} + 6N + 1 & 16 - a_{22} & a_{23} - (6N - 1) \\ -a_{21} & a_{22} & -a_{23} \\ -(6N + 1) & -16 & (6N - 1) \end{pmatrix}, \quad \begin{aligned} (2N - 1)^3 &= a_{22}(6N - 1) - 16a_{23}, \\ (2N + 1)^3 &= a_{22}(6N + 1) + 16a_{21}. \end{aligned} \quad (3.2)$$

Two Eqs. (3.2) need that at least two of a_{2j} be polynomials in N of the 2nd degree, e.g., a_{21} and a_{22} are quadratic polynomials. To balance the cubic degrees in (3.2) choose a_{2j} as polynomials on residue class of N modulo t_3 which will be found later, i.e., $N = t_3m + j$, $0 \leq j < t_3$,

$$a_{21} = r_2m^2 + r_1m + r_0, \quad a_{22} = k_2m^2 + k_1m + k_0, \quad a_{23} = l_1m + l_0. \quad (3.3)$$

Substitute (3.3) into (3.2) and obtain

$$\begin{aligned} k_2 = r_2 &= \frac{4t_3^2}{3}, \quad k_1 = \frac{8t_3}{9}(3j - 2), \quad 3t_3k_0 + 4(r_1 - l_1) = \frac{t_3}{3}(9 + 16j + 12j^2), \\ (6j + 1)k_0 + 16r_0 &= (2j + 1)^3, \quad (6j - 1)k_0 - 16l_0 = (2j - 1)^3, \quad \frac{r_1 + l_1}{1 + 12j} = \frac{2t_3}{9}. \end{aligned}$$

The minimal value of t_3 , providing $k_1 \in \mathbb{Z}_+$ in the above equalities be integer, is $t_3 = 9$, that leads to $k_1 = 8(3j - 2)$ and $k_2 = r_2 = 108$. The other five parameters, $r_0(j), r_1(j), k_0(j), l_0(j)$ and $l_1(j)$ may be found if we find numerically matrix \mathcal{A}_3 for nine first semigroups R_{18+2j}^3 , $0 \leq j \leq 8$.

1. $n = 18m$, $m \geq 1$,

$$\begin{pmatrix} 108m^2 + 55m + 1 & -(108m^2 - 16m - 15) & -(53m - 1) \\ -m(108m + 1) & 108m^2 - 16m + 1 & -m \\ -(54m + 1) & -16 & 54m - 1 \end{pmatrix},$$

$$G = 314928m^5 + 110808m^4 + 16632m^3 + 532m^2 - 62m,$$

$$F = 629856m^5 + 215784m^4 + 34020m^3 + 1890m^2 - 109m - 1, \quad m \leq 15,$$

$$F = 629856m^5 + 221616m^4 - 58320m^3 + 1944m^2 - 108m - 1, \quad m \geq 16.$$

2. $n = 18m + 2$, $m \geq 1$; $n \neq 2$

$$\begin{pmatrix} 108m^2 + 37m + 3 & -(108m^2 + 8m - 3) & -(11m + 1) \\ -(108m^2 - 17m - 4) & 108m^2 + 8m + 13 & -(43m + 4) \\ -(54m + 7) & -16 & 54m + 5 \end{pmatrix},$$

$$G = 314928m^5 + 285768m^4 + 104760m^3 + 15244m^2 + 786m + 6,$$

$$F = 629856m^5 + 571536m^4 + 190512m^3 + 31752m^2 + 2548m + 75.$$

3. $n = 18m + 4, m \geq 1; \quad n \neq 4.$

$$\begin{pmatrix} 108m^2 + 55m + 7 & -(108m^2 + 32m + 1) & -(5m + 1) \\ -(108m^2 + m - 6) & 108m^2 + 32m + 17 & -(49m + 10) \\ -(54m + 13) & -16 & 54m + 11 \end{pmatrix},$$

$$G = 314928m^5 + 460728m^4 + 270648m^3 + 77476m^2 + 10674m + 564,$$

$$F = 629856m^5 + 921456m^4 + 532656m^3 + 153144m^2 + 21812m + 1223.$$

4. $n = 18m + 6, m \geq 1; \quad n \neq 6.$

$$\begin{pmatrix} 108m^2 + 109m + 25 & -(108m^2 + 56m - 3) & -(35m + 11) \\ -(108m^2 + 55m + 6) & 108m^2 + 56m + 13 & -(19m + 6) \\ -(54m + 19) & -16 & 54m + 17 \end{pmatrix},$$

$$G = 314928m^5 + 635688m^4 + 514296m^3 + 202780m^2 + 38434m + 2778,$$

$$F = 629856m^5 + 1271376m^4 + 968112m^3 + 355752m^2 + 63828m + 4499.$$

5. $n = 18m + 8, m \geq 0,$

$$\begin{pmatrix} 108m^2 + 145m + 44 & -(108m^2 + 80m + 1) & -(47m + 20) \\ -(108m^2 + 91m + 19) & 108m^2 + 80m + 17 & -(7m + 3) \\ -(54m + 25) & -16 & 54m + 23 \end{pmatrix},$$

$$G = 314928m^5 + 810648m^4 + 835704m^3 + 428308m^2 + 108658m + 10888,$$

$$F_{m=0,1} = 629856m^5 + 1580472m^4 + 1604772m^3 + 821178m^2 + 210979m + 21700,$$

$$F_{m \geq 2} = 629856m^5 + 1621296m^4 + 1590192m^3 + 753624m^2 + 173908m + 15695.$$

6. $n = 18m + 10, m \geq 0,$

$$\begin{pmatrix} 108m^2 + 163m + 58 & -(108m^2 + 104m + 13) & -(41m + 22) \\ -(108m^2 + 109m + 27) & 108m^2 + 104m + 29 & -(13m + 7) \\ -(54m + 31) & -16 & 54m + 29 \end{pmatrix},$$

$$G = 314928m^5 + 985608m^4 + 1234872m^3 + 769612m^2 + 237666m + 29022,$$

$$F = 55222, \quad m = 0,$$

$$F_{m \geq 1} = 629856m^5 + 1971216m^4 + 2398896m^3 + 1429704m^2 + 419252m + 48539.$$

7. $n = 18m + 12, m \geq 0,$

$$\begin{pmatrix} 108m^2 + 163m + 61 & -(108m^2 + 128m + 33) & -(17m + 11) \\ -(108m^2 + 109m + 24) & 108m^2 + 128m + 49 & -(37m + 24) \\ -(54m + 37) & -16 & 54m + 35 \end{pmatrix},$$

$$G = 314928m^5 + 1160568m^4 + 1711800m^3 + 1257796m^2 + 459058m + 66444,$$

$$F = 629856m^5 + 2321136m^4 + 3394224m^3 + 2466936m^2 + 892404m + 128663.$$

8. $n = 18m + 14$, $m \geq 0$,

$$\begin{pmatrix} 108m^2 + 199m + 90 & -(108m^2 + 152m + 45) & -(29m + 22) \\ -(108m^2 + 145m + 47) & 108m^2 + 152m + 61 & -(25m + 19) \\ -(54m + 43) & -16 & 54m + 41 \end{pmatrix},$$

$$G = 314928m^5 + 1335528m^4 + 2266488m^3 + 1917916m^2 + 807378m + 135042,$$

$$F = 629856m^5 + 2671056m^4 + 4482864m^3 + 3730536m^2 + 1541908m + 253539.$$

9. $n = 18m + 16$, $m \geq 0$,

$$\begin{pmatrix} 108m^2 + 217m + 108 & -(108m^2 + 176m + 65) & -(23m + 20) \\ -(108m^2 + 163m + 59) & 108m^2 + 176m + 81 & -(31m + 27) \\ -(54m + 49) & -16 & 54m + 47 \end{pmatrix},$$

$$G = 314928m^5 + 1510488m^4 + 2898936m^3 + 2776756m^2 + 1325266m + 251824,$$

$$F = 629856m^5 + 3020976m^4 + 5758128m^3 + 5458968m^2 + 2576660m + 484767.$$

3.2 NUMERICAL SEMIGROUPS R_n^3 , $n \equiv 1 \pmod{2}$

The matrix of minimal relations \mathcal{A}_3 reads,

$$\begin{pmatrix} a_{21} + (3N + 2) & 8 - a_{22} & a_{23} - (3N + 1) \\ -a_{21} & a_{22} & -a_{23} \\ -(3N + 2) & -8 & 3N + 1 \end{pmatrix}, \quad \begin{aligned} (2N)^3 &= a_{22}(3N + 1) - 8a_{23}, \\ (2N + 2)^3 &= a_{22}(3N + 2) + 8a_{21}. \end{aligned}$$

We skip intermediate calculations repeating the procedure performed in section 3.1.

1. $n = 18m + 1$, $m \geq 0$,

$$\begin{pmatrix} 216m^2 + 29m + 1 & -(216m^2 - 8m) & -m \\ -(216m^2 + 2m - 1) & 216m^2 - 8m + 8 & -(26m + 1) \\ -(27m + 2) & -8 & 27m + 1 \end{pmatrix},$$

$$G = 629856m^5 + 160380m^4 + 23220m^3 + 2330m^2 + 73m,$$

$$F = 1259712m^5 + 320760m^4 + 44712m^3 + 4644m^2 + 154m - 1.$$

2. $n = 18m + 3$, $m \geq 1$; $n \neq 3$.

$$\begin{pmatrix} 216m^2 + 83m + 8 & -(216m^2 + 40m) & -(7m + 1) \\ -(216m^2 + 56m + 3) & 216m^2 + 40m + 8 & -(20m + 3) \\ -(27m + 5) & -8 & 27m + 4 \end{pmatrix},$$

$$G = 629856m^5 + 510300m^4 + 172260m^3 + 30134m^2 + 2657m + 91,$$

$$F = 181, \quad m = 0,$$

$$F = 1259712m^5 + 1020600m^4 + 332424m^3 + 55404m^2 + 4698m + 157, \quad m \geq 1.$$

3. $n = 18m + 5, m \geq 0,$

$$\begin{pmatrix} 216m^2 + 128m + 19 & -(216m^2 + 88m + 8) & -(4m + 1) \\ -(216m^2 + 101m + 11) & 216m^2 + 88m + 16 & -(23m + 6) \\ -(27m + 8) & -8 & 27m + 7 \end{pmatrix},$$

$$G = 629856m^5 + 860220m^4 + 476820m^3 + 134186m^2 + 18993m + 1066,$$

$$F = 1259712m^5 + 1720440m^4 + 946728m^3 + 263412m^2 + 37082m + 2107.$$

4. $n = 18m + 7, m \geq 0,$

$$\begin{pmatrix} 216m^2 + 191m + 42 & -(216m^2 + 136m + 16) & -(19m + 7) \\ -(216m^2 + 164m + 31) & 216m^2 + 136m + 24 & -(8m + 3) \\ -(27m + 11) & -8 & 27m + 10 \end{pmatrix},$$

$$G = 629856m^5 + 1210140m^4 + 936900m^3 + 365246m^2 + 71609m + 5637,$$

$$F = 10745, \quad m = 0,$$

$$F_{m \geq 1} = 1259712m^5 + 2420280m^4 + 1840968m^3 + 693468m^2 + 129322m + 9537.$$

5. $n = 18m + 9, m \geq 0,$

$$\begin{pmatrix} 216m^2 + 245m + 69 & -(216m^2 + 184m + 32) & -(25m + 12) \\ -(216m^2 + 218m + 55) & 216m^2 + 184m + 40 & -(2m + 1) \\ -(27m + 14) & -8 & 27m + 13 \end{pmatrix},$$

$$G = 629856m^5 + 1560060m^4 + 1552500m^3 + 776234m^2 + 195001m + 19684,$$

$$F_{m \leq 3} = 1259712m^5 + 3108456m^4 + 3083184m^3 + 1537596m^2 + 385666m + 38919,$$

$$F_{m \geq 4} = 1259712m^5 + 3120120m^4 + 3061800m^3 + 1488132m^2 + 358074m + 34087.$$

6. $n = 18m + 11, m \geq 0,$

$$\begin{pmatrix} 216m^2 + 290m + 97 & -(216m^2 + 232m + 56) & -(22m + 13) \\ -(216m^2 + 263m + 80) & 216m^2 + 232m + 64 & -(5m + 3) \\ -(27m + 17) & -8 & 27m + 16 \end{pmatrix},$$

$$\begin{aligned}
G &= 629856m^5 + 1909980m^4 + 2323620m^3 + 1417694m^2 + 433745m + 53223, \\
F &= 103589, \quad m = 0, \\
F_{m \geq 1} &= 1259712m^5 + 3819960m^4 + 4609224m^3 + 2766636m^2 + 826058m + 98125.
\end{aligned}$$

7. $n = 18m + 13$, $m \geq 0$,

$$\begin{pmatrix} 216m^2 + 326m + 123 & -(216m^2 + 280m + 90) & -(10m + 7) \\ -(216m^2 + 299m + 103) & 216m^2 + 280m + 96 & -(17m + 12) \\ -(27m + 20) & -8 & 27m + 19 \end{pmatrix},$$

$$\begin{aligned}
G &= 629856m^5 + 2259900m^4 + 3250260m^3 + 2342114m^2 + 845465m + 122286, \\
F &= 1259712m^5 + 3120120m^4 + 3061800m^3 + 1488132m^2 + 358074m + 34087.
\end{aligned}$$

8. $n = 18m + 15$, $m \geq 0$,

$$\begin{pmatrix} 216m^2 + 380m + 167 & -(216m^2 + 328m + 120) & -(16m + 13) \\ -(216m^2 + 353m + 144) & 216m^2 + 328m + 128 & -(11m + 9) \\ -(27m + 23) & -8 & 27m + 22 \end{pmatrix},$$

$$\begin{aligned}
G &= 629856m^5 + 2609820m^4 + 4332420m^3 + 3601550m^2 + 1499153m + 249937, \\
F &= 1259712m^5 + 5219640m^4 + 8637192m^3 + 7135452m^2 + 2943162m + 484897.
\end{aligned}$$

9. $n = 18m + 17$, $m \geq 0$,

$$\begin{pmatrix} 216m^2 + 425m + 209 & -(216m^2 + 376m + 160) & -(13m + 12) \\ -(216m^2 + 398m + 183) & 216m^2 + 376m + 168 & -(14m + 13) \\ -(27m + 26) & -8 & 27m + 25 \end{pmatrix},$$

$$\begin{aligned}
G &= 629856m^5 + 2959740m^4 + 5570100m^3 + 5247626m^2 + 2474713m + 467304, \\
F &= 1259712m^5 + 5919480m^4 + 11117736m^3 + 10433124m^2 + 4892186m + 917039.
\end{aligned}$$

Formulas for $F(R_n^3)$ in this section coincide with those obtained in [8] if $b_{22} > b_{11}$.

3.3 EXCEPTIONAL SEMIGROUPS R_n^3 , $n = 4, 6$

There exist 3 exceptional semigroups (1 symmetric $R_3^3 \equiv \langle 2^3, 3^3 \rangle$ and 2 nonsymmetric R_4^3, R_6^3) with minimal relations which do not obey the matrix Reps presented in section 3.1, 3.2. Below we give Reps of two nonsymmetric semigroups.

$$R_4^3 : \begin{pmatrix} 7 & -1 & -1 \\ -1 & 18 & -9 \\ -6 & -17 & 10 \end{pmatrix}, \quad \begin{matrix} G = 558 \\ F = 1098 \end{matrix}, \quad R_6^3 : \begin{pmatrix} 31 & -10 & -5 \\ -6 & 13 & -6 \\ -25 & -3 & 11 \end{pmatrix}, \quad \begin{matrix} G = 2670 \\ F = 5249 \end{matrix}.$$

4 NUMERICAL SEMIGROUPS R_n^4 , $n \geq 4$

These semigroups were not studied in [8], however using a weak argumentation its authors predict that $F(R_n^4)$ is given by polynomial expressions in n on residue class of n modulo 88 *'whereas experimental tests make us believe that we need 40 formulas'* [8].

Assuming that the matrix of minimal relations comprise the polynomial expressions in n on residue class of n modulo 40, we have calculated numerically these matrices in two different cases $n = 0, 1 \pmod{2}$. In section 4.4 we give also expressions for genus of semigroups R_{20m+9}^4 and R_{20m-9}^4 which illustrate the forthcoming Theorem 2.

4.1 NUMERICAL SEMIGROUPS R_n^4 , $n = 0 \pmod{2}$

1. $n = 40m$, $m \geq 4$; $n \neq 40, 80, 120$.

$$\begin{pmatrix} 8160m^2 + 161m + 1 & -(320m^2 - 1280m + 1) & -(7840m^2 - 159m + 1) \\ -(160m^2 + m) & 320m^2 + 1 & -(160m^2 - m) \\ -(8000m^2 + 160m + 1) & -1280m & 8000m^2 - 160m + 1 \end{pmatrix}$$

2. $n = 40m + 2$, $m \geq 1$; $n \neq 2$.

$$\begin{pmatrix} 4640m^2 + 526m + 15 & -(4480m^2 + 64m + 6) & -(160m^2 - 18m - 1) \\ -(4400m^2 + 495m + 14) & 4800m^2 + 160m + 11 & -(400m^2 + 65m + 2) \\ -(240m^2 + 31m + 1) & -(320m^2 + 96m + 5) & 560m^2 + 47m + 1 \end{pmatrix}$$

3. $n = 40m + 4$, $m \geq 0$.

$$\begin{pmatrix} 2880m^2 + 616m + 33 & -8(320m^2 + 32m + 1) & -(320m^2 + 40m + 1) \\ -(2240m^2 + 473m + 25) & 2880m^2 + 448m + 25 & -(640m^2 + 135m + 7) \\ -(640m^2 + 143m + 8) & -(320m^2 + 192m + 17) & 960m^2 + 175m + 8 \end{pmatrix}$$

4. $n = 40m + 6$, $m \geq 1$; $n \neq 6$.

$$\begin{pmatrix} 2800m^2 + 885m + 70 & -(1600m^2 + 160m - 7) & -(1200m^2 + 325m + 22) \\ -(1760m^2 + 550m + 43) & 1920m^2 + 448m + 30 & -(160m^2 + 58m + 5) \\ -(1040m^2 + 335m + 27) & -(320m^2 + 288m + 37) & 1360m^2 + 383m + 27 \end{pmatrix}$$

5. $n = 40m + 8$, $m \geq 0$.

$$\begin{pmatrix} 2240m^2 + 932m + 97 & -4(320m^2 + 64m + 1) & -(960m^2 + 356m + 33) \\ -(800m^2 + 325m + 33) & 1600m^2 + 640m + 69 & -(800m^2 + 315m + 31) \\ -(1440m^2 + 607m + 64) & -(320m^2 + 384m + 65) & 1760m^2 + 671m + 64 \end{pmatrix}$$

6. $n = 40m + 10$, $m \geq 1$; $n \neq 10$.

$$\begin{pmatrix} 2480m^2 + 1283m + 166 & -(960m^2 + 160m - 17) & -(1520m^2 + 723m + 86) \\ -(640m^2 + 324m + 41) & 1280m^2 + 640m + 84 & -(640m^2 + 316m + 39) \\ -(1840m^2 + 959m + 125) & -(320m^2 + 480m + 101) & 2160m^2 + 1039m + 125 \end{pmatrix}$$

7. $n = 40m + 12$, $m \geq 0$.

$$\begin{pmatrix} 1280m^2 + 787m + 121 & -(960m^2 + 448m + 51) & -(320m^2 + 179m + 25) \\ -(320m^2 + 183m + 26) & 2240m^2 + 1472m + 247 & -(1920m^2 + 1129m + 166) \\ -(960m^2 + 604m + 95) & -(1280m^2 + 1024m + 196) & 2240m^2 + 1308m + 191 \end{pmatrix}$$

8. $n = 40m + 14$, $m \geq 1$; $n \neq 14$.

$$\begin{pmatrix} 2720m^2 + 1954m + 351 & -(640m^2 + 64m - 54) & -(2080m^2 + 1410m + 239) \\ -(80m^2 + 51m + 8) & 960m^2 + 736m + 143 & -(880m^2 + 605m + 104) \\ -(2640m^2 + 1903m + 343) & -(320m^2 + 672m + 197) & 2960m^2 + 2015m + 343 \end{pmatrix}$$

9. $n = 40m + 16$, $m \geq 0$.

$$\begin{pmatrix} 1920m^2 + 1570m + 321 & -(640m^2 + 256m + 2) & -(1280m^2 + 994m + 193) \\ -(800m^2 + 645m + 130) & 1600m^2 + 1280m + 261 & -(800m^2 + 635m + 126) \\ -(1120m^2 + 925m + 191) & -(960m^2 + 1024m + 259) & 2080m^2 + 1629m + 319 \end{pmatrix}$$

10. $n = 40m + 18$, $m \geq 0$.

$$\begin{pmatrix} 1120m^2 + 1026m + 235 & -(640m^2 + 448m + 74) & -(480m^2 + 418m + 91) \\ -(1040m^2 + 937m + 211) & 2880m^2 + 2656m + 621 & -(1840m^2 + 1639m + 365) \\ -(80m^2 + 89m + 24) & -(2240m^2 + 2208m + 547) & 2320m^2 + 2057m + 456 \end{pmatrix}$$

11. $n = 40m + 20$, $m \geq 2$; $n \neq 60$.

$$\begin{pmatrix} 4160m^2 + 4241m + 1081 & -(320m^2 - 320m - 239) & -(3840m^2 + 3761m + 921) \\ -(320m^2 + 322m + 81) & 640m^2 + 640m + 162 & -(320m^2 + 318m + 79) \\ -(3840m^2 + 3919m + 1000) & -(320m^2 + 960m + 401) & 4160m^2 + 4079m + 1000 \end{pmatrix}$$

12. $n = 40m - 18$, $m \geq 1$.

$$\begin{pmatrix} 2320m^2 - 2057m + 456 & -(2240m^2 - 2208m + 547) & -(80m^2 - 89m + 24) \\ -(1840m^2 - 1639m + 365) & 2880m^2 - 2656m + 621 & -(1040m^2 - 937m + 211) \\ -(480m^2 - 418m + 91) & -(640m^2 - 448m + 74) & 1120m^2 - 1026m + 235 \end{pmatrix}$$

13. $n = 40m - 16$, $m \geq 1$.

$$\begin{pmatrix} 2080m^2 - 1629m + 319 & -(960m^2 - 1024m + 259) & -(1120m^2 - 925m + 191) \\ -(800m^2 - 635m + 126) & 1600m^2 - 1280m + 261 & -(800m^2 - 645m + 130) \\ -(1280m^2 - 994m + 193) & -(640m^2 - 256m + 2) & 1920m^2 - 1570m + 321 \end{pmatrix}$$

14. $n = 40m - 14$, $m \geq 2$; $n \neq 26$.

$$\begin{pmatrix} 2960m^2 - 2015m + 343 & -(320m^2 - 672m + 197) & -(2640m^2 - 1903m + 343) \\ -(880m^2 - 605m + 104) & 960m^2 - 736m + 143 & -(80m^2 - 51m + 8) \\ -(2080m^2 - 1410m + 239) & -(640m^2 - 64m - 54) & 2720m^2 - 1954m + 351 \end{pmatrix}$$

15. $n = 40m - 12, m \geq 1$.

$$\begin{pmatrix} 2240m^2 - 1308m + 191 & -4(320m^2 - 256m + 49) & -(960m^2 - 604m + 95) \\ -(1920m^2 - 1129m + 166) & 2240m^2 - 1472m + 247 & -(320m^2 - 183m + 26) \\ -(320m^2 - 179m + 25) & -(960m^2 - 448m + 51) & 1280m^2 - 787m + 121 \end{pmatrix}$$

16. $n = 40m - 10, m \geq 2; \quad n \neq 30$.

$$\begin{pmatrix} 2160m^2 - 1039m + 125 & -(320m^2 - 480m + 101) & -(1840m^2 - 959m + 125) \\ -(640m^2 - 316m + 39) & 1280m^2 - 640m + 84 & -(640m^2 - 324m + 41) \\ -(1520m^2 - 723m + 86) & -(960m^2 - 160m - 17) & 2480m^2 - 1283m + 166 \end{pmatrix}$$

18. $n = 40m - 8, m \geq 1$.

$$\begin{pmatrix} 1760m^2 - 671m + 64 & -(320m^2 - 384m + 65) & -(1440m^2 - 607m + 64) \\ -(800m^2 - 315m + 31) & 1600m^2 - 640m + 69 & -(800m^2 - 325m + 33) \\ -(960m^2 - 356m + 33) & -(1280m^2 - 256m + 4) & 2240m^2 - 932m + 97 \end{pmatrix}$$

19. $n = 40m - 6, m \geq 1$.

$$\begin{pmatrix} 1360m^2 - 383m + 27 & -(320m^2 - 288m + 37) & -(1040m^2 - 335m + 27) \\ -(160m^2 - 58m + 5) & 1920m^2 - 448m + 30 & -(1760m^2 - 550m + 43) \\ -(1200m^2 - 325m + 22) & -(1600m^2 - 160m - 7) & 2800m^2 - 885m + 70 \end{pmatrix}$$

20. $n = 40m - 4, m \geq 1$.

$$\begin{pmatrix} 960m^2 - 175m + 8 & -(320m^2 - 192m + 17) & -(640m^2 - 143m + 8) \\ -(640m^2 - 135m + 7) & 2880m^2 - 448m + 25 & -(2240m^2 - 473m + 25) \\ -(320m^2 - 40m + 1) & -(2560m^2 - 256m + 8) & 2880m^2 - 616m + 33 \end{pmatrix}$$

21. $n = 40m - 2, m \geq 1$.

$$\begin{pmatrix} 560m^2 - 47m + 1 & -(320m^2 - 96m + 5) & -(240m^2 - 31m + 1) \\ -(400m^2 - 65m + 2) & 4800m^2 - 160m + 11 & -(4400m^2 - 495m + 14) \\ -(160m^2 + 18m - 1) & -(4480m^2 - 64m + 6) & 4640m^2 - 526m + 15 \end{pmatrix}.$$

4.2 NUMERICAL SEMIGROUPS $R_n^4, n \equiv 1 \pmod{2}$

1. $n = 20m + 1, m \geq 1$.

$$\begin{pmatrix} 230m^2 + 30m + 1 & -32m(5m - 1) & -2m(35m + 1) \\ -10m(15m + 1) & 16(50m^2 + 10m + 1) & -(650m^2 + 50m + 1) \\ -(80m^2 + 20m + 1) & -16(40m^2 + 12m + 1) & 720m^2 + 52m + 1 \end{pmatrix}$$

2. $n = 20m + 3, m \geq 1; \quad n \neq 3$.

$$\begin{pmatrix} 530m^2 + 178m + 15 & -16(10m^2 - 6m - 1) & -(370m^2 + 94m + 6) \\ -(60m^2 + 16m + 1) & 16(20m^2 + 8m + 1) & -(260m^2 + 72m + 5) \\ -(470m^2 + 162m + 14) & -32(5m^2 + 7m + 1) & 630m^2 + 166m + 11 \end{pmatrix}$$

3. $n = 20m + 5, m \geq 1; \quad n \neq 5.$

$$\begin{pmatrix} 490m^2 + 258m + 34 & -16(30m^2 + 10m + 1) & -(10m^2 - 2m - 1) \\ -(320m^2 + 164m + 21) & 16(40m^2 + 20m + 3) & -(320m^2 + 156m + 19) \\ -(170m^2 + 94m + 13) & -32(5m^2 + 5m + 1) & 330m^2 + 154m + 18 \end{pmatrix}$$

4. $n = 20m + 7, m \geq 2; \quad n \neq 7, 27.$

$$\begin{pmatrix} 2(565m^2 + 417m + 77) & -32(5m^2 - 7m - 3) & -(970m^2 + 638m + 105) \\ -(130m^2 + 94m + 17) & 16(10m^2 + 6m + 1) & -(30m^2 + 22m + 4) \\ -(1000m^2 + 740m + 137) & -16(20m + 7) & 1000m^2 + 660m + 109 \end{pmatrix}$$

5. $n = 20m + 9, m \geq 1; \quad n \neq 9.$

$$\begin{pmatrix} 430m^2 + 402m + 94 & -16(10m^2 + 2m - 1) & -(270m^2 + 230m + 49) \\ -(290m^2 + 266m + 61) & 32(15m^2 + 13m + 3) & -(190m^2 + 170m + 38) \\ -(140m^2 + 136m + 33) & -16(20m^2 + 24m + 7) & 460m^2 + 400m + 87 \end{pmatrix}$$

6. $n = 20m - 9, m \geq 1.$

$$\begin{pmatrix} 460m^2 - 400m + 87 & -16(20m^2 - 24m + 7) & -(140m^2 - 136m + 33) \\ -(190m^2 - 170m + 38) & 32(15m^2 - 13m + 3) & -(290m^2 - 266m + 61) \\ -(270m^2 - 230m + 49) & -16(10m^2 - 2m - 1) & 430m^2 - 402m + 94 \end{pmatrix}$$

7. $n = 20m - 7, m \geq 3; \quad n \neq 13, 33.$

$$\begin{pmatrix} 1030m^2 - 682m + 113 & -32(5m^2 - 13m + 4) & -(870m^2 - 646m + 120) \\ -(30m^2 - 22m + 4) & 16(10m^2 - 6m + 1) & -(130m^2 - 94m + 17) \\ -(1000m^2 - 660m + 109) & -16(20m - 7) & 1000m^2 - 740m + 137 \end{pmatrix}$$

8. $n = 20m - 5, m \geq 1.$

$$\begin{pmatrix} 330m^2 - 154m + 18 & -32(5m^2 - 5m + 1) & -(170m^2 - 94m + 13) \\ -(320m^2 - 156m + 19) & 16(40m^2 - 20m + 3) & -(320m^2 - 164m + 21) \\ -(10m^2 + 2m - 1) & -16(30m^2 - 10m + 1) & 490m^2 - 258m + 34 \end{pmatrix}$$

9. $n = 20m - 3, m \geq 2; \quad n \neq 17.$

$$\begin{pmatrix} 630m^2 - 166m + 11 & -32(5m^2 - 7m + 1) & -(470m^2 - 162m + 14) \\ -(260m^2 - 72m + 5) & 16(20m^2 - 8m + 1) & -(60m^2 - 16m + 1) \\ -(370m^2 - 94m + 6) & -16(10m^2 + 6m - 1) & 530m^2 - 178m + 15 \end{pmatrix}$$

10. $n = 20m - 1, m \geq 1.$

$$\begin{pmatrix} 720m^2 - 52m + 1 & -16(40m^2 - 12m + 1) & -(80m^2 - 20m + 1) \\ -(650m^2 - 50m + 1) & 16(50m^2 - 10m + 1) & -10m(15m - 1) \\ -2m(35m - 1) & -32m(5m + 1) & 230m^2 - 30m + 1 \end{pmatrix}.$$

4.3 EXCEPTIONAL SEMIGROUPS R_n^4

There exist 18 exceptional semigroups R_n^4 (3 symmetric R_3^4 , R_5^4 , R_7^4 and 15 nonsymmetric) with minimal relations which do not obey the matrix Reps presented in sections 4.1, 4.2. We give them below.

$$\begin{aligned}
R_3^4 &\equiv \langle 2^4, 3^4 \rangle, & G &= 560, & F &= 1199, & R_7^4 &: \begin{pmatrix} 256 & 0 & -81 \\ -17 & 16 & -4 \\ -256 & 0 & 81 \end{pmatrix}, & G &= 181200, & F &= 362399 \\
R_5^4 &: \begin{pmatrix} 81 & 0 & -16 \\ -34 & 16 & -1 \\ -81 & 0 & 16 \end{pmatrix}, & G &= 14280, & F &= 28559, & R_6^4 &: \begin{pmatrix} 113 & -23 & -17 \\ -43 & 30 & -5 \\ -70 & -7 & 22 \end{pmatrix}, & G &= 41713, & F &= 78308. \\
R_9^4 &: \begin{pmatrix} 155 & -80 & -11 \\ -61 & 96 & -38 \\ -94 & -16 & 49 \end{pmatrix}, & G &= 502480, & F &= 994223. \\
R_{10}^4 &: \begin{pmatrix} 207 & -67 & -47 \\ -41 & 84 & -39 \\ -166 & -17 & 86 \end{pmatrix}, & G &= 965342, & F &= 1897924. \\
R_{13}^4 &: \begin{pmatrix} 485 & -32 & -238 \\ -12 & 80 & -53 \\ -473 & -48 & 291 \end{pmatrix}, & G &= 6071192, & F &= 12005295. \\
R_{14}^4 &: \begin{pmatrix} 359 & -89 & -135 \\ -8 & 143 & -104 \\ -351 & -54 & 239 \end{pmatrix}, & G &= 7729559, & F &= 15400797. \\
R_{17}^4 &: \begin{pmatrix} 668 & -176 & -277 \\ -193 & 208 & -45 \\ -475 & -32 & 322 \end{pmatrix}, & G &= 24979344, & F &= 48247935. \\
R_{20}^4 &: \begin{pmatrix} 1243 & -85 & -763 \\ -81 & 162 & -79 \\ -1162 & -77 & 842 \end{pmatrix}, & G &= 90813516, & F &= 176868200. \\
R_{26}^4 &: \begin{pmatrix} 1667 & -212 & -1043 \\ -379 & 367 & -37 \\ -1288 & -155 & 1080 \end{pmatrix}, & G &= 365363593, & F &= 720624113. \\
R_{27}^4 &: \begin{pmatrix} 2359 & -112 & -1657 \\ -241 & 272 & -561 \\ -2118 & -160 & 1713 \end{pmatrix}, & G &= 647256024, & F &= 1230618127.
\end{aligned}$$

$$\begin{aligned}
R_{30}^4 : \begin{pmatrix} 1609 & -665 & -649 \\ -363 & 724 & -357 \\ -1246 & -59 & 1006 \end{pmatrix}, \quad & \begin{aligned} G &= 739585479, \\ F &= 1465271324. \end{aligned} \\
R_{33}^4 : \begin{pmatrix} 2949 & -400 & -1959 \\ -80 & 464 & -349 \\ -2869 & -64 & 2308 \end{pmatrix}, \quad & \begin{aligned} G &= 1782545568, \\ F &= 3555061055. \end{aligned} \\
R_{40}^4 : \begin{pmatrix} 8805 & -4 & -7205 \\ -161 & 321 & -159 \\ -8644 & -317 & 7364 \end{pmatrix}, \quad & \begin{aligned} G &= 10589583194, \\ F &= 21173668803. \end{aligned} \\
R_{60}^4 : \begin{pmatrix} 10205 & -1203 & -7805 \\ -723 & 1442 & -717 \\ -9482 & -239 & 8522 \end{pmatrix}, \quad & \begin{aligned} G &= 67447447193, \\ F &= 133546213800. \end{aligned} \\
R_{80}^4 : \begin{pmatrix} 33605 & -2 & -30405 \\ -642 & 1281 & -638 \\ -32963 & -1279 & 31043 \end{pmatrix}, \quad & \begin{aligned} G &= 680612207996, \\ F &= 1361182355203. \end{aligned} \\
R_{120}^4 : \begin{pmatrix} 75367 & -1922 & -68647 \\ -1443 & 2881 & -1437 \\ -73924 & -959 & 70084 \end{pmatrix}, \quad & \begin{aligned} G &= 7758023870871, \\ F &= 15421051483202. \end{aligned}
\end{aligned}$$

4.4 DUALITY OF SEMIGROUPS $R_{T_4 m - k}^4$ AND $R_{T_4 m + k}^4$, $T_4 = 40$

A careful observation of matrices of minimal relations for semigroups $R_{T_4 m \pm k}^4$, $T_4 = 40$, and their genera allows to prove two statements.

Theorem 1 *Let two semigroups be given by their minimal relations (1.9),*

$$R_{T_4 m \pm k}^4 : \begin{pmatrix} E_{11}^{\pm}(m) & -E_{12}^{\pm}(m) & -E_{13}^{\pm}(m) \\ -E_{21}^{\pm}(m) & E_{22}^{\pm}(m) & -E_{23}^{\pm}(m) \\ -E_{31}^{\pm}(m) & -E_{32}^{\pm}(m) & E_{33}^{\pm}(m) \end{pmatrix}, \quad k \leq \frac{T_4}{2}, \quad (4.1)$$

where $E_{ij}^-(m)$ and $E_{ij}^+(m)$ are given by polynomials

$$E_{ij}^-(m) = A_{ij}^- m^2 - B_{ij}^- m + C_{ij}^-, \quad E_{ij}^+(m) = A_{ij}^+ m^2 + B_{ij}^+ m + C_{ij}^+. \quad (4.2)$$

Then the following duality relations hold

$$\begin{aligned}
A_{11}^+ &= A_{33}^-, & B_{11}^+ &= B_{33}^-, & C_{11}^+ &= C_{33}^-, & A_{22}^+ &= A_{22}^-, & B_{22}^+ &= B_{22}^-, & C_{22}^+ &= C_{22}^-, \\
A_{33}^+ &= A_{11}^-, & B_{33}^+ &= B_{11}^-, & C_{33}^+ &= C_{11}^-, & A_{12}^+ &= A_{32}^-, & B_{12}^+ &= B_{32}^-, & C_{12}^+ &= C_{32}^-, \\
A_{21}^+ &= A_{23}^-, & B_{21}^+ &= B_{23}^-, & C_{21}^+ &= C_{23}^-, & A_{31}^+ &= A_{13}^-, & B_{31}^+ &= B_{13}^-, & C_{31}^+ &= C_{13}^-, \\
A_{13}^+ &= A_{31}^-, & B_{13}^+ &= B_{31}^-, & C_{13}^+ &= C_{31}^-, & A_{23}^+ &= A_{21}^-, & B_{23}^+ &= B_{21}^-, & C_{23}^+ &= C_{21}^-, \\
A_{32}^+ &= A_{12}^-, & B_{32}^+ &= B_{12}^-, & C_{32}^+ &= C_{12}^-.
\end{aligned}$$

Proof According to formulas in sections 4.1, 4.2 consider the polynomial Rep of minimal relations for numerical semigroups $R_{T_4m-k}^4$,

$$\begin{aligned} E_{11}^-(m)(T_4m-k-1)^4 &= E_{12}^-(m)(T_4m-k)^4 &+& E_{13}^-(m)(T_4m-k+1)^4, \\ E_{22}^-(m)(T_4m-k)^4 &= E_{21}^-(m)(T_4m-k-1)^4 &+& E_{23}^-(m)(T_4m-k+1)^4, \\ E_{33}^-(m)(T_4m-k+1)^4 &= E_{31}^-(m)(T_4m-k-1)^4 &+& E_{32}^-(m)(T_4m-k)^4, \end{aligned} \quad (4.3)$$

and R_{40m+k}^4 ,

$$\begin{aligned} E_{11}^+(m)(T_4m+k-1)^4 &= E_{12}^+(m)(T_4m+k)^4 &+& E_{13}^+(m)(T_4m+k+1)^4, \\ E_{22}^+(m)(T_4m+k)^4 &= E_{21}^+(m)(T_4m+k-1)^4 &+& E_{23}^+(m)(T_4m+k+1)^4, \\ E_{33}^+(m)(T_4m+k+1)^4 &= E_{31}^+(m)(T_4m+k-1)^4 &+& E_{32}^+(m)(T_4m+k)^4, \end{aligned} \quad (4.4)$$

where $E_{ij}^-(m)$ and $E_{ij}^+(m)$ are defined in (4.2). Replacing $m \rightarrow -m$ in (4.3) we get

$$\begin{aligned} E_{11}^-(-m)(T_4m+k+1)^4 &= E_{12}^-(-m)(T_4m+k)^4 &+& E_{13}^-(-m)(T_4m+k-1)^4, \\ E_{22}^-(-m)(T_4m+k)^4 &= E_{21}^-(-m)(T_4m+k+1)^4 &+& E_{23}^-(-m)(T_4m+k-1)^4, \\ E_{33}^-(-m)(T_4m+k-1)^4 &= E_{31}^-(-m)(T_4m+k+1)^4 &+& E_{32}^-(-m)(T_4m+k)^4. \end{aligned}$$

Compare the last three Eqs. with (4.4). Both Reps coincide for arbitrary m iff

$$\begin{aligned} E_{11}^+(m) &= E_{33}^-(-m), & E_{12}^+(m) &= E_{32}^-(-m), & E_{13}^+(m) &= E_{31}^-(-m), \\ E_{22}^+(m) &= E_{22}^-(-m), & E_{21}^+(m) &= E_{23}^-(-m), & E_{23}^+(m) &= E_{21}^-(-m), \\ E_{33}^+(m) &= E_{11}^-(-m), & E_{31}^+(m) &= E_{13}^-(-m), & E_{32}^+(m) &= E_{12}^-(-m). \end{aligned} \quad (4.5)$$

Substituting (4.2) into (4.5) we arrive at the proof of Theorem.

The next Theorem is motivated by systematic calculation on genera of semigroups $R_{T_4m+k}^4$ and $R_{T_4m-k}^4$. We illustrate it by the following example,

$$\begin{aligned} G(R_{20m+9}^4) &= 8(7766000m^6 + 21049600m^5 + 23809780m^4 + 14385560m^3 + \\ &\quad 4895936m^2 + 889781m + 67446), \\ G(R_{20m-9}^4) &= 8(7766000m^6 - 21049600m^5 + 23809780m^4 - 14385560m^3 + \\ &\quad 4895936m^2 - 889781m + 67446). \end{aligned}$$

Theorem 2 Let two numerical semigroups $R_{T_4m \mp k}^4$ be given by their minimal relations (4.1) and let their genera $G^\pm(m)$ are given by formulas,

$$R_{T_4m-k}^4 : G^-(m) = \sum_{r=0}^6 g_r^- m^r \quad R_{T_4m+k}^4 : G^+(m) = \sum_{r=0}^6 g_r^+ m^r. \quad (4.6)$$

Then $g_{2r}^- = g_{2r}^+$ and $g_{2r+1}^- = -g_{2r+1}^+$.

Proof Write formulas (1.10) of D_k , $0 \leq k \leq 3$, for two numerical semigroups: $R_{T_4m-k}^4$

$$\begin{aligned} D_0^-(m) &= E_{11}^-(m) E_{22}^-(m) E_{33}^-(m), & D_1^-(m) &= E_{12}^-(m) E_{23}^-(m) E_{31}^-(m), \\ D_2^-(m) &= E_{13}^-(m) E_{32}^-(m) E_{21}^-(m), \\ D_3^-(m) &= (T_4m - k - 1)^4 + (T_4m - k)^4 + (T_4m - k + 1)^4, \end{aligned}$$

and R_{40m+k}^4 ,

$$\begin{aligned} D_0^+(m) &= E_{11}^+(m) E_{22}^+(m) E_{33}^+(m), & D_1^+(m) &= E_{12}^+(m) E_{23}^+(m) E_{31}^+(m), \\ D_2^+(m) &= E_{13}^+(m) E_{32}^+(m) E_{21}^+(m), \\ D_3^+(m) &= (T_4m + k - 1)^4 + (T_4m + k)^4 + (T_4m + k + 1)^4. \end{aligned}$$

Combining the above formulas with (4.5) we obtain

$$D_0^-(m) = D_0^+(-m), \quad D_1^-(m) = D_2^+(-m), \quad D_2^-(m) = D_1^+(-m), \quad D_3^-(m) = D_3^+(-m).$$

that together with genus definition in (1.9) gives $G^-(m) = G^+(-m)$. Combining the last equality with (4.6) we arrive at the proof of Theorem.

5 CONCLUDING REMARKS

In the present section we state a conjecture and put a question devoted to numerical semigroups R_n^k , $n > 3, k \geq 5$, where an appearance of symmetric semigroups R_n^k seems very rare. Numerical calculations give only two semigroups R_5^{11} and R_5^{13} among others R_{2p+1}^k , $2 \leq p \leq 50, 5 \leq k \leq 10^3$.

Conjecture 1 *Let a numerical semigroup R_n^k , $n = T_k m + j$, be given by their minimal relations on residue class of n modulo T_k ,*

$$R_{T_k m + j}^k : \begin{pmatrix} E_{11}^{(k)}(m) & -E_{12}^{(k)}(m) & -E_{13}^{(k)}(m) \\ -E_{21}^{(k)}(m) & E_{22}^{(k)}(m) & -E_{23}^{(k)}(m) \\ -E_{31}^{(k)}(m) & -E_{32}^{(k)}(m) & E_{33}^{(k)}(m) \end{pmatrix}, \quad j \leq \frac{T_k}{2}. \quad (5.1)$$

If $k = 2q$, then polynomial $E_{ij}^{(k)}(m)$ reads,

$$E_{ij}^{(2q)}(m) = A_{ij}m^q + B_{ij}m^{q-1} + \dots + C_{ij}m + D_{ij}, \quad 1 \leq i, j \leq 3, \quad (5.2)$$

and the Frobenius number and genus have the asymptotics: $F(n), G(n) = \mathcal{O}(n^{3q})$.

If $k = 2q + 1$, then the matrix elements with $(i, j) = (1, 1), (1, 2), (2, 1), (2, 2)$ are given by

$$E_{ij}^{(2q+1)}(m) = K_{ij}m^{q+1} + I_{ij}m^q + \dots + J_{ij}m + H_{ij}, \quad (5.3)$$

while the matrix elements with $(i, j) = (1, 3), (2, 3), (3, 1), (3, 2), (3, 3)$ read

$$E_{ij}^{(2q+1)}(m) = M_{ij}m^q + N_{ij}m^{q-1} + \dots + P_{ij}m + S_{ij}, \quad (5.4)$$

and the Frobenius number and genus have the asymptotics: $F(n), G(n) = \mathcal{O}(n^{3q+2})$.

Question 1 Keeping in mind $T_2 = 4$, $T_3 = 18$, $T_4 = 40$, find T_k for $k \geq 5$.

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A PROOF OF PROPOSITIONS

Proof of Proposition 2. Semigroup R_3^3 is symmetric due to requirement (1.4a). Find more n which satisfy (1.4a),

$$(n+1)^3 = e_1(n-1)^3 + e_2n^3, \quad e_1, e_2 \in \mathbb{N}, \quad n > 3.$$

Simplifying the last equality we obtain the Diophantine Eq.

$$\begin{aligned} (e_1 + e_2 - 1)t^3 + 3(2e_1 + 3e_2 - 4)t^2 + 3(4e_1 + 9e_2 - 16)t + \\ 8e_1 + 27e_2 - 64 = 0, \quad t = n - 3. \end{aligned} \quad (\text{A.1})$$

Decompose the whole integer lattice $\mathbb{Z}_2^+ := \{e_1, e_2 \mid e_1, e_2 \geq 1\}$ in different sets,

$$\begin{aligned} \mathbb{Z}_2^+ &= \bigcup_{j=1}^5 \mathbb{E}_j, \quad \mathbb{E}_1 = \{e_1, e_2 \mid e_1 \geq 5; e_2 = 1\}, \quad \mathbb{E}_2 = \{e_1, e_2 \mid e_1 \geq 2; e_2 = 2\}, \\ \mathbb{E}_3 &= \{e_1, e_2 \mid e_1 \geq 1; e_2 \geq 3\}, \quad \mathbb{E}_4 = \{e_1, e_2 \mid 1 \leq e_1 \leq 4; e_2 = 1\}, \\ \mathbb{E}_5 &= \{e_1 = e_2 = 1\}. \end{aligned}$$

If $(e_1, e_2) \in \mathbb{E}_j$, $1 \leq j \leq 3$, then the sequence of coefficients in Eq. (1.8) has no changes of signs and therefore, by Descartes' rule of signs, Eq. (1.8) has no positive solutions in t . If $(e_1, e_2) \in \mathbb{E}_j$, $j = 4, 5$, then a straightforward numerical verification shows that neither of 5 cubic Eqs. (A.1) has integer positive solution in t .

Consider an alternative way to symmetrize R_n^3 by providing condition (1.4b), which may occur only when $n = 2q + 1$ and results in the Diophantine Eq. in $c_1, c_2 \in \mathbb{N}$, $q > 1$, $(2q + 1)^3 = c_1q^3 + c_2(q + 1)^3$, i.e.,

$$(c_1 + c_2 - 8)q^3 + 3(c_2 - 4)q^2 + 3(c_2 - 2)q + c_2 - 1 = 0. \quad (\text{A.2})$$

Substituting $q = p + 1$, $p > 0$, into (A.2) we obtain the cubic Diophantine Eq. in p ,

$$(c_1 + c_2 - 8)p^3 + 3(c_1 + 2c_2 - 12)p^2 + 3(c_1 + 4c_2 - 18)p + c_1 + 8c_2 - 27 = 0, \quad (\text{A.3})$$

which has no positive integer solutions p . Indeed, to prove this statement, we make use of Descartes' rule of signs for integer coefficients in Eq. (A.3). For this purpose decompose the whole integer lattice $\mathbb{Z}_2^+ := \{c_1, c_2 \mid c_1, c_2 \geq 1\}$ as follows,

$$\mathbb{Z}_2^+ = \mathbb{C} \cup \overline{\mathbb{C}}, \quad \mathbb{C} = \bigcup_{j=1}^7 \mathbb{C}_j, \quad \overline{\mathbb{C}} = \bigcup_{j=1}^6 \overline{\mathbb{C}}_j, \quad \text{where}$$

$$\mathbb{C}_1 = \{c_1, c_2 \mid 1 \leq c_1 \leq 7, c_1 \geq 19; c_2 = 1\},$$

$$\mathbb{C}_2 = \{c_1, c_2 \mid 1 \leq c_1 \leq 6, c_1 \geq 11; c_2 = 2\},$$

$$\mathbb{C}_3 = \{c_1, c_2 \mid 1 \leq c_1 \leq 3, c_1 \geq 6; c_2 = 3\}, \quad \mathbb{C}_4 = \{c_1, c_2 \mid c_1 \geq 4; c_2 = 4\},$$

$$\mathbb{C}_5 = \{c_1, c_2 \mid c_1 \geq 3; c_2 = 5\}, \quad \mathbb{C}_6 = \{c_1, c_2 \mid c_1 \geq 2; c_2 = 6\},$$

$$\mathbb{C}_7 = \{c_1, c_2 \mid c_1 \geq 1; c_2 \geq 7\},$$

$$\overline{\mathbb{C}}_1 = \{c_1, c_2 \mid 8 \leq c_1 \leq 18; c_2 = 1\}, \quad \overline{\mathbb{C}}_2 = \{c_1, c_2 \mid 7 \leq c_1 \leq 10; c_2 = 2\},$$

$$\overline{\mathbb{C}}_3 = \{c_1, c_2 \mid 4 \leq c_1 \leq 5; c_2 = 3\}, \quad \overline{\mathbb{C}}_4 = \{c_1, c_2 \mid 1 \leq c_1 \leq 3; c_2 = 4\},$$

$$\overline{\mathbb{C}}_5 = \{c_1, c_2 \mid 1 \leq c_1 \leq 2; c_2 = 5\}, \quad \overline{\mathbb{C}}_6 = \{c_1 = 1; c_2 = 6\}.$$

If $(c_1, c_2) \in \mathbb{C}$ then the sequence of coefficients in Eq. (A.3) has no changes of signs and therefore, by Descartes' rule of signs, Eq. (A.3) has no positive solutions in p . Regarding the rest of the cases, when $(c_1, c_2) \in \overline{\mathbb{C}}$, a straightforward numerical verification shows that neither of 23 cubic Eqs. (A.3) has integer positive solution in p . \square

Proof of Proposition 3. Semigroup R_3^4 is symmetric due to requirement (1.4a). Find more n which satisfy (1.4a),

$$(n+1)^4 = f_1(n-1)^4 + f_2 n^4, \quad f_1, f_2 \in \mathbb{N}, \quad n > 4.$$

Simplify the last equality and obtain the Diophantine Eq.

$$(f_1 + f_2 - 1)u^4 + 4(2f_1 + 3f_2 - 4)u^3 + 6(4f_1 + 9f_2 - 16)u^2 + \quad (\text{A.4})$$

$$4(8f_1 + 27f_2 - 64)u + 16f_1 + 81f_2 - 256 = 0, \quad u = n - 3.$$

Decompose the whole integer lattice $\mathbb{Z}_2^+ := \{f_1, f_2 \mid f_1, f_2 \geq 1\}$ in different sets, $\mathbb{Z}_2^+ = \bigcup_{j=1}^5 \mathbb{F}_j$, where

$$\mathbb{F}_1 = \{f_1, f_2 \mid f_1 \geq 11; f_2 = 1\}, \quad \mathbb{F}_2 = \{f_1, f_2 \mid f_1 \geq 6; f_2 = 2\},$$

$$\mathbb{F}_3 = \{f_1, f_2 \mid f_1 \geq 1; f_2 \geq 3\}, \quad \mathbb{F}_4 = \{f_1, f_2 \mid 1 \leq f_1 \leq 10; f_2 = 1\},$$

$$\mathbb{F}_5 = \{f_1, f_2 \mid 1 \leq f_1 \leq 5; f_2 = 2\}.$$

If $(f_1, f_2) \in \mathbb{F}_j$, $1 \leq j \leq 3$, then the sequence of coefficients in Eq. (A.4) has no changes of signs and therefore, by Descartes' rule of signs, Eq. (A.4) has no positive solutions in u . In the rest of the cases, when $(f_1, f_2) \in \mathbb{F}_j$, $j = 4, 5$, a straightforward numerical verification shows that neither of 15 quartic Eqs. (A.4) has integer positive solution in u .

Consider another way to find symmetric semigroups R_n^4 by providing condition (1.4b), which may occur only when $n = 2q + 1$ and results in the Diophantine Eq. in $h_1, h_2 \in \mathbb{N}$, $q > 1$,

$$(2q + 1)^4 = h_1 q^4 + h_2 (q + 1)^4. \quad (\text{A.5})$$

Equation (A.5) has solutions $q = 2, h_1 = 34, h_2 = 1$ and $q = 3, h_1 = 17, h_2 = 4$, which correspond to symmetric semigroups R_5^4 and R_7^4 , respectively. We show that Eq. (A.5) has no more positive integer solutions. Denote $v = q - 2$, $v > 0$, and represent (A.5) as follows,

$$(h_1 + h_2 - 16)v^4 + 4(2h_1 + 3h_2 - 40)v^3 + 6(4h_1 + 9h_2 - 100)v^2 + 4(8h_1 + 27h_2 - 250)v + 16h_1 + 81h_2 - 625 = 0. \quad (\text{A.6})$$

Decompose the whole integer lattice $\mathbb{Z}_2^+ := \{h_1, h_2 \mid h_1, h_2 \geq 1\}$ in different sets,

$$\mathbb{Z}_2^+ = \mathbb{H} \cup \overline{\mathbb{H}}, \quad \mathbb{H} = \bigcup_{j=1}^{15} \mathbb{H}_j, \quad \overline{\mathbb{H}} = \bigcup_{j=0}^{14} \overline{\mathbb{H}}_j, \quad \text{where}$$

$$\mathbb{H}_1 = \{h_1, h_2 \mid 1 \leq h_1 \leq 15, h_1 \geq 34; h_2 = 1\},$$

$$\mathbb{H}_2 = \{h_1, h_2 \mid 1 \leq h_1 \leq 14, h_1 \geq 29; h_2 = 2\},$$

$$\mathbb{H}_3 = \{h_1, h_2 \mid 1 \leq h_1 \leq 13, h_1 \geq 24; h_2 = 3\},$$

$$\mathbb{H}_4 = \{h_1, h_2 \mid 1 \leq h_1 \leq 12, h_1 \geq 19; h_2 = 4\},$$

$$\mathbb{H}_5 = \{h_1, h_2 \mid 1 \leq h_1 \leq 11, h_1 \geq 15; h_2 = 5\},$$

$$\mathbb{H}_6 = \{h_1, h_2 \mid 1 \leq h_1 \leq 8, h_1 \geq 12; h_2 = 6\},$$

$$\mathbb{H}_7 = \{h_1, h_2 \mid 1 \leq h_1 \leq 3, h_1 \geq 10; h_2 = 7\},$$

$$\mathbb{H}_j = \{h_1, h_2 \mid h_1 \geq 16 - j; h_2 = j\}, \quad 8 \leq j \leq 14,$$

$$\mathbb{H}_{15} = \{h_1, h_2 \mid h_1 \geq 1; h_2 \geq 15\},$$

$$\overline{\mathbb{H}}_1 = \{h_1, h_2 \mid 16 \leq h_1 \leq 33; h_2 = 1\}, \quad \overline{\mathbb{H}}_2 = \{h_1, h_2 \mid 15 \leq h_1 \leq 28; h_2 = 2\},$$

$$\overline{\mathbb{H}}_3 = \{h_1, h_2 \mid 14 \leq h_1 \leq 23; h_2 = 3\}, \quad \overline{\mathbb{H}}_4 = \{h_1, h_2 \mid 13 \leq h_1 \leq 18; h_2 = 4\},$$

$$\overline{\mathbb{H}}_5 = \{h_1, h_2 \mid 12 \leq h_1 \leq 14; h_2 = 5\}, \quad \overline{\mathbb{H}}_6 = \{h_1, h_2 \mid 9 \leq h_1 \leq 11; h_2 = 6\},$$

$$\overline{\mathbb{H}}_7 = \{h_1, h_2 \mid 4 \leq h_1 \leq 9; h_2 = 7\},$$

$$\overline{\mathbb{H}}_j = \{h_1, h_2 \mid 1 \leq h_1 \leq 15 - j; h_2 = j\}, \quad 8 \leq j \leq 14.$$

If $(h_1, h_2) \in \mathbb{H}$ then the sequence of coefficients in Eq. (A.6) has no changes of signs and therefore, by Descartes' rule of signs, Eq. (A.6) has no positive solutions in p . Regarding the rest of the cases, when $(h_1, h_2) \in \overline{\mathbb{H}}$, a straightforward numerical verification shows that neither of 88 quartic Eqs. (A.6) has integer positive solution in v . \square

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